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Abstract
Among the reductions of the resonant three-wave interaction system to six-dimensional differential systems, one of them has been specifically mentioned as being linked to the generic sixth Painlevé equation P6. We derive this link explicitly, and we establish the connection to a three-degree-of-freedom Hamiltonian previously considered for P6.

PACS numbers: 02.30.Hq, 02.30.Ik

1. Introduction

The three-wave resonant interaction system (3WRI) in $1 + 1$ dimensions, i.e. whose impulsions k_j and pulsations ω_j have a zero sum, $k_1 + k_2 + k_3 = 0$, $\omega_1 + \omega_2 + \omega_3 = 0$, can be mathematically described by six coupled first-order partial differential equations (PDEs) in six dependent complex variables $u_j, \bar{u}_j$ (the amplitudes) and two independent variables x, t [21],

$$\begin{cases} u_{j,t} + c_j u_{j,x} - i\bar{u}_k\bar{u}_l = 0, \\ \bar{u}_{j,t} + c_j \bar{u}_{j,x} + iu_k u_l = 0, \qquad i^2 = -1, \end{cases} \tag{1}$$

in which (j, k, l) denotes any permutation of $(1, 2, 3)$ and c_j are the constant values of the group velocities, with $(c_2 - c_3)(c_3 - c_1)(c_1 - c_2) \neq 0$.

This system admits a third-order Lax pair [21]. In the traceless zero-curvature representation, this is given by [1]

$$\rho = -\frac{c_3 - c_1}{c_2 - c_3}, \qquad \sigma = \frac{c_1 - c_2}{c_3 - c_1}, \tag{2}$$

$$L = \frac{i\lambda}{c_1 - c_2} \begin{pmatrix} -1 + 2\rho & 0 & 0 \\ 0 & 2 - \rho & 0 \\ 0 & 0 & -1 - \rho \end{pmatrix} + \frac{i}{c_1 - c_2} \begin{pmatrix} 0 & -\sigma\rho u_3 & \sigma\rho \bar{u}_2 \\ \sigma \bar{u}_3 & 0 & -\sigma u_1 \\ -u_2 & -\bar{u}_1 & 0 \end{pmatrix}, \tag{3}$$

$$M = \frac{i\lambda}{c_1 - c_2} \begin{pmatrix} c_1 - 2c_2\rho & 0 & 0 \\ 0 & -2c_1 + c_2\rho & 0 \\ 0 & 0 & c_1 + c_2\rho \end{pmatrix}$$

$$+ \frac{i}{c_1 - c_2} \begin{pmatrix} 0 & c_3\sigma\rho u_3 & -c_2\sigma\rho \bar{u}_2 \\ -c_3\sigma \bar{u}_3 & 0 & c_1\sigma u_1 \\ c_2 u_2 & c_1 \bar{u}_1 & 0 \end{pmatrix}, \tag{4}$$

$$[\partial_x - L, \partial_t - M] = 0, \tag{5}$$

in which λ, the spectral parameter, is an arbitrary complex constant.

The purpose of this paper is to show the existence of at least one noncharacteristic one-dimensional reduction to a system of ordinary differential equations (ODEs) integrable with the generic sixth Painlevé function and to integrate it explicitly. Indeed, at the present time, various reductions of this system have been integrated with most of the six Painlevé functions [9, 15, 17], but no explicit link with the generic sixth Painlevé equation has been found up to now.

Since a noncharacteristic reduction preserves the order, it is necessary, in order to integrate with P6 which depends on four parameters α, β, γ, δ, that the reduced system of ODEs depends on two arbitrary parameters. The determination of all subgroups of the invariance group of the 3WRI system, which allows one to generate all the classical reductions, has been performed in [17].

The paper is organized as follows. After recalling in section 2 the singularity structure of the system, we define the reduction in section 3, and then generate its first integrals from the Lax pair in section 4. The explicit integration with the generic P6 equation is performed in sections 5 and 6. In section 7, we discuss the link with two previous works on the same kind of third-order matrix Lax pair and the possible implications on a second-order matrix Lax pair for P6.

2. Local singularity analysis

The singularity structure analysis of the 3WRI system (1) has been performed in the more general setting of three space variables [11]. The result is a unique family of movable singularities, in which the six components u_j, $\bar{u}_j$ all behave like a simple pole

$$u_j \sim a_j X^{-1}, \qquad \bar{u}_j \sim b_j X^{-1}, \tag{6}$$

in the neighbourhood of a singular manifold [20],

$$\varphi(x, t) - \varphi_0 = 0, \tag{7}$$

in which φ is an arbitrary function of the independent variables, φ_0 is an arbitrary movable constant and the expansion variable $X(x, t)$ [6] vanishes along with $\varphi - \varphi_0$ and satisfies

$$X_x = 1 + O(X), \qquad X_t = -C + O(X), \qquad C = -\frac{\varphi_t}{\varphi_x}. \tag{8}$$

The linearized system of (1) in the neighbourhood of the expansion (6) is of Fuchsian type near the singular manifold, and its six Fuchs indices r are $r = -1, 0, 0, 2, 2, 3$.

The existence of the isospectral Lax pair (5) implies that no movable logarithms can enter the expansion (6), which is indeed the case [11].

For any noncharacteristic reduction, the resulting system of ODEs also admits the above family of movable simple poles, and the first integrals can only have the singularity degrees 2, 2, 3.

3. A noncharacteristic reduction and its Lax pair

The problem of finding a noncharacteristic reduction of the 3WRI system and its Lax pair has been tackled by several authors using different theoretical frameworks. Among them, Kitaev [15] gave the one-dimensional reduction (already given in [9] in the restricted case $\beta_j = 0$)

$$
\begin{cases}
\zeta = \dfrac{x}{t}, & \beta_1 + \beta_2 + \beta_3 = 0, \\
u_j(x, t) = (t(c_j - \zeta))^{-1+i\beta_j} \psi_j, \\
\bar{u}_j(x, t) = (t(c_j - \zeta))^{-1-i\beta_j} \bar{\psi}_j,
\end{cases}
\tag{9}
$$

in which β_j are constants, to the six first-order ODEs

$$
\begin{cases}
\dfrac{d}{d\zeta} \psi_j = i(c_j - \zeta)^{-i\beta_j} (c_k - \zeta)^{-1-i\beta_k} (c_l - \zeta)^{-1-i\beta_l} \bar{\psi}_k \bar{\psi}_l, \\
\dfrac{d}{d\zeta} \bar{\psi}_j = -i(c_j - \zeta)^{i\beta_j} (c_k - \zeta)^{-1+i\beta_k} (c_l - \zeta)^{-1+i\beta_l} \psi_k \psi_l,
\end{cases}
\tag{10}
$$

in which (j, k, l) denotes any permutation of $(1, 2, 3)$. However, he performed the integration only in a particular case.

As noted by Kitaev, ζ, c_1, c_2, c_3 only contribute by their cross-ratio, so this system depends only on two parameters β_j. It is nevertheless advisable to keep c_j s to display the ternary symmetry.

To compute the reduced Lax pair, let us represent the PDE Lax pair as the 1-form

$$
\omega = L\varphi \, dx + M\varphi \, dt,
\tag{11}
$$

in which (L, M) depends on (x, t, λ). One wants to find two operators $\mathcal{L}, \mathcal{M}$ and one scalar variable μ, so as to represent the reduced Lax pair as

$$
\Omega = \mathcal{L}\Phi \, d\zeta + \mathcal{M}\Phi \, d\mu,
\tag{12}
$$

in which $(\mathcal{L}, \mathcal{M})$ depends on (ζ, μ).

One first eliminates $u_j, \bar{u}_j, x, dx$ from the reduction (9) to obtain

$$
\omega = L_1(\zeta, \lambda, t)\varphi \, d\zeta + M_1(\zeta, \lambda, t)\varphi \, dt,
\tag{13}
$$

then one applies a change of basis

$$
\varphi = P\Phi,
\tag{14}
$$

in which the transition matrix P is chosen to depend only on t, so as to gather the dependence on (t, λ) into a single variable μ. This matrix takes the form

$$
P = \text{diag}(t^{i(\beta_3 - \beta_2)/3}, t^{i(\beta_1 - \beta_3)/3}, t^{i(\beta_2 - \beta_1)/3}),
\tag{15}
$$

$$
\Omega = P^{-1}\omega - P^{-1}(dP)P^{-1}\varphi,
\tag{16}
$$

and the result is (12), with $\mu = \lambda t$.

The reduced traceless Lax pair $(\mathcal{L}, \mathcal{M})$ in zero-curvature representation

$$
[\partial_\zeta - \mathcal{L}, \partial_\mu - \mathcal{M}] = 0
\tag{17}
$$

depends on the constant spectral parameter μ,

$$\mathcal{L} = \frac{i}{c_1 - c_2} \mu \begin{pmatrix} -1 + 2\rho & 0 & 0 \\ 0 & 2 - \rho & 0 \\ 0 & 0 & -1 - \rho \end{pmatrix}$$

$$+ \frac{i}{c_1 - c_2} \begin{pmatrix} 0 & -\sigma\rho\psi_3(c_3 - \zeta)^{-1+\beta_3 i} & \sigma\rho\bar\psi_2(c_2 - \zeta)^{-1-\beta_2 i} \\ \sigma\bar\psi_3(c_3 - \zeta)^{-1-\beta_3 i} & 0 & -\sigma\psi_1(c_1 - \zeta)^{-1+\beta_1 i} \\ -\psi_2(c_2 - \zeta)^{-1+\beta_2 i} & -\bar\psi_1(c_1 - \zeta)^{-1-\beta_1 i} & 0 \end{pmatrix}, \quad (18)$$

$$\mathcal{M} = \frac{i}{c_1 - c_2} \begin{pmatrix} c_1 - \zeta - 2\rho(c_2 - \zeta) & 0 & 0 \\ 0 & -2(c_1 - \zeta) + \rho(c_2 - \zeta) & 0 \\ 0 & 0 & c_1 - \zeta + \rho(c_2 - \zeta) \end{pmatrix}$$

$$+ \frac{i}{3}\mu^{-1} \begin{pmatrix} \beta_2 - \beta_3 & 0 & 0 \\ 0 & \beta_3 - \beta_1 & 0 \\ 0 & 0 & \beta_1 - \beta_2 \end{pmatrix}$$

$$- \frac{i}{c_1 - c_2}\mu^{-1} \begin{pmatrix} 0 & -\sigma\rho\psi_3(c_3 - \zeta)^{\beta_3 i} & \sigma\rho\bar\psi_2(c_2 - \zeta)^{-\beta_2 i} \\ \sigma\bar\psi_3(c_3 - \zeta)^{-\beta_3 i} & 0 & -\sigma\psi_1(c_1 - \zeta)^{\beta_1 i} \\ -\psi_2(c_2 - \zeta)^{\beta_2 i} & -\bar\psi_1(c_1 - \zeta)^{-\beta_1 i} & 0 \end{pmatrix}. \quad (19)$$

The singularities of the matrix $\mathcal{M}$ in the complex spectral parameter are $\mu = 0$ (of the Fuchsian type) and $\mu = \infty$ (of the non-Fuchsian type).

4. The two first integrals and the reduced fourth-order system

The presence of one Fuchsian singularity in the monodromy matrix $\mathcal{M}$ allows one to generate easily the first integrals. Indeed, denoting by $\mathcal{M}_{-1}$ the residue of the matrix $\mathcal{M}$ at the Fuchsian singularity $\mu = 0$,

$$\mathcal{M} = \mathcal{M}_{-1}\mu^{-1} + \mathcal{M}_0, \quad (20)$$

the invariants of the residue $\mathcal{M}_{-1}$ are constants of the motion. These are generated by the characteristic polynomial

$$\det(\mathcal{M}_{-1} - z) = -z^3 - \left(K_1 + \frac{\beta_1^2 + \beta_2^2 + \beta_3^2}{6}\right)z + 2i\left(K_2 - \frac{(\beta_2 - \beta_3)(\beta_3 - \beta_1)(\beta_1 - \beta_2)}{54}\right), \quad (21)$$

in which K_1, K_2 denote the only two first integrals

$$K_1 = [(c_2 - c_3)\psi_1\bar\psi_1 + (c_3 - c_1)\psi_2\bar\psi_2 + (c_1 - c_2)\psi_3\bar\psi_3]((c_2 - c_3)(c_3 - c_1)(c_1 - c_2))^{-1}, \quad (22)$$

$$\begin{aligned} K_2 = \Big[&\tfrac{1}{2}\psi_1\psi_2\psi_3(c_1 - \zeta)^{\beta_1 i}(c_2 - \zeta)^{\beta_2 i}(c_3 - \zeta)^{\beta_3 i} \\ &+ \tfrac{1}{2}\bar\psi_1\bar\psi_2\bar\psi_3(c_1 - \zeta)^{-\beta_1 i}(c_2 - \zeta)^{-\beta_2 i}(c_3 - \zeta)^{-\beta_3 i} + \tfrac{1}{6}((\beta_2 - \beta_3)(c_2 - c_3)\psi_1\bar\psi_1 \\ &+ (\beta_3 - \beta_1)(c_3 - c_1)\psi_2\bar\psi_2 + (\beta_1 - \beta_2)(c_1 - c_2)\psi_3\bar\psi_3)\Big] \\ &\times ((c_2 - c_3)(c_3 - c_1)(c_1 - c_2))^{-1}. \end{aligned} \quad (23)$$

These two first integrals have the singularity degrees 2 and 3, in agreement with the results of section 2. The two first integrals allow us to reduce the order from 6 to 4. Introducing the six variables ρ_j, φ_j,

$$\psi_j = \rho_j\,e^{i\varphi_j}, \qquad \bar\psi_j = \rho_j\,e^{-i\varphi_j}, \quad (24)$$

the two invariants only depend on the four variables ρ_j, χ,

$$K_1 = \left[(c_2 - c_3)\rho_1^2 + (c_3 - c_1)\rho_2^2 + (c_1 - c_2)\rho_3^2\right]((c_1 - c_2)(c_2 - c_3)(c_3 - c_1))^{-1}, \tag{25}$$

$$K_2 = \left[\rho_1\rho_2\rho_3 \cos\chi + \frac{1}{6}\sum_j(\beta_k - \beta_l)(c_k - c_l)\rho_j^2\right]((c_1 - c_2)(c_2 - c_3)(c_3 - c_1))^{-1}, \tag{26}$$

$$\chi = \sum_j(\varphi_j + \beta_j \log(c_j - \zeta)). \tag{27}$$

Therefore, the differential system for ρ_j, χ is closed. This allows one to discard the three variables φ_j, remembering only their first derivatives

$$\varphi_j' = \frac{\rho_k\rho_l}{(c_k - \zeta)(c_l - \zeta)\rho_j}\cos\chi, \tag{28}$$

and to focus on the closed fourth-order system

$$\rho_j' = \frac{\rho_k\rho_l}{(c_k - \zeta)(c_l - \zeta)}\sin\chi, \tag{29}$$

$$\chi' = \sum_j\left(\frac{\rho_k\rho_l}{(c_k - \zeta)(c_l - \zeta)\rho_j}\cos\chi - \frac{\beta_j}{c_j - \zeta}\right), \tag{30}$$

which admits the two first integrals (25) and (26).

In order to integrate the system (29)–(30), it is advisable to lower the order from 4 to 2, for instance by building a single second-order ODE depending on the four parameters K_1, K_2, β_j.

5. Link to a classified second-order, second-degree ODE

Following the procedure of [17], we derive the change of variables which allows the fourth-order system (29)–(30) to be explicitly integrated in terms of the generic P6 equation.

Given any two components ρ_j^2, they admit a unique (up to a multiplicative factor) linear combination Y whose first derivative has no contribution from $\sin\chi$, e.g. [17, equation (5.41)],

$$\begin{cases} Y = \dfrac{c_3 - \zeta}{c_2 - \zeta}\rho_2^2 - \rho_3^2, \\[2mm] Y' = -\dfrac{c_2 - c_3}{(c_2 - \zeta)^2}\rho_2^2, \\[2mm] Y'' = -2\dfrac{c_2 - c_3}{(c_2 - \zeta)^3}\left[\rho_2^2 + \dfrac{c_2 - \zeta}{(c_1 - \zeta)(c_3 - \zeta)}\rho_1\rho_2\rho_3 \sin\chi\right]. \end{cases} \tag{31}$$

By eliminating ρ_j and χ between Y, Y', Y'' and the two invariants, one builds an ODE for Y [17, equation (5.42)], which has second order, second degree and the binomial type

$$Y''^2 = F(Y', Y, \zeta). \tag{32}$$

This binomial type has been 'classified' [8], i.e. all such ODEs with the Painlevé property have been enumerated and integrated. Therefore, if the present ODE for $Y(x)$ has the Painlevé property, there should exist a homographic transformation mapping it to one such classified ODE. This is indeed the case, and there exists an affine transformation

$$Y = \frac{c_2 - c_3}{c_2 - \zeta}[\lambda(\zeta)y(x) + \mu(\zeta)], \qquad x = X(\zeta), \tag{33}$$

which maps the ODE for $Y(\zeta)$ to the canonical ODE SD.I.a for $y(x)$ [8, equation (5.4)],

$$-x^2(x-1)^2 y''^2 - 4y'(xy'-y)^2 + 4y'^2(xy'-y)$$

$$+ A_0 y'^2 + A_2(xy'-y) + \left(A_3 + \frac{A_0^2}{4}\right) y' + A_4 = 0. \tag{34}$$

Among the equations determining the three functions $\lambda(\zeta)$, $\mu(\zeta)$, $X(\zeta)$, the leading ones are

$$\begin{cases} \lambda'' = 0, \qquad \mu'' = 0, \qquad (\lambda^2 X')' = 0, \\ \dfrac{X'}{X(X-1)} - \dfrac{\lambda}{(c_1-\zeta)(c_2-\zeta)(c_3-\zeta)} = 0, \end{cases} \tag{35}$$

and this results in six possible values for the three functions $\lambda(\zeta)$, $\mu(\zeta)$, $X(\zeta)$,

$$\lambda = (c_j - c_k)(c_l - \zeta), \qquad x = -\frac{(c_j - c_l)(c_k - \zeta)}{(c_k - c_j)(c_l - \zeta)}, \tag{36}$$

in which (j, k, l) is any permutation of $(1, 2, 3)$. Let us choose for instance the value

$$\lambda = -(c_3 - c_1)(c_2 - \zeta), \tag{37}$$

$$x = -\frac{(c_1 - c_2)(c_3 - \zeta)}{(c_3 - c_1)(c_2 - \zeta)}, \tag{38}$$

$$\mu = -\frac{K_1}{2}(c_3 - c_1)(c_2 - \zeta) + \frac{\beta_2}{4}[\beta_1(c_3 - c_1)(c_2 - \zeta) - \beta_2(c_1 - c_2)(c_3 - \zeta)]. \tag{39}$$

The three variables ρ_j^2 are then linear in y' and y,

$$\rho_j^2 = \frac{(c_1 - c_2)(c_2 - c_3)(c_j - \zeta)}{c_2 - \zeta} y' - (c_3 - c_1)(c_j - c_2)y$$

$$- (c_j - c_k)(c_j - c_l)(1 - \delta_{j,2})\frac{K_1}{2} + (c_j - c_k)(c_j - c_l)\frac{\beta_2 \beta_{4-j}}{4}, \tag{40}$$

in which (j, k, l) is any permutation of $(1, 2, 3)$, and the link with the four constants in SD.I.a is

$$\begin{cases} A_0 = -2K_1 - \dfrac{1}{2}\left(\beta_1^2 + \beta_2^2 + \beta_3^2\right), \\[2mm] A_2 = \beta_2\left[\dfrac{\beta_3 - \beta_1}{3}K_1 + 4K_2 + \dfrac{1}{4}\beta_2^2(\beta_3 - \beta_1)\right], \\[2mm] A_3 = \beta_3\left[\dfrac{\beta_1 - \beta_2}{3}K_1 + 4K_2 + \dfrac{1}{4}\beta_3^2(\beta_1 - \beta_2)\right], \\[2mm] A_4 = -4K_2^2 - \dfrac{5\beta_2^2 + 2\beta_2\beta_3 + 2\beta_3^2}{18}K_1^2 - \dfrac{2(\beta_3 - \beta_1)}{3}K_1 K_2 \\[2mm] \qquad - \beta_2^2\dfrac{5\beta_2^2 + 8\beta_2\beta_3 + 8\beta_3^2}{24}K_1 - \beta_2^2(\beta_3 - \beta_1)K_2 - \beta_2^4\dfrac{\beta_2^2 + 3\beta_2\beta_3 + 3\beta_3^2}{16}. \end{cases} \tag{41}$$

In order to display the permutation symmetry, it is convenient to introduce the additional constant

$$A_1 = \beta_1\left[\frac{\beta_2 - \beta_3}{3}K_1 + 4K_2 + \frac{1}{4}\beta_1^2(\beta_2 - \beta_3)\right]. \tag{42}$$

Equation SD.I.a, first derived by Chazy [5, equation B-V, p 340] up to some homographic transformation, has been integrated by Bureau *et al* [4], and its general solution is an algebraic

transform of the generic P6 equation for $u(x)$,

$$P6 : u'' = \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right]u'$$
$$+ \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right], \tag{43}$$

$(2\alpha, -2\beta, 2\gamma, 1-2\delta) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2)$.

The formulae in [4] have been further simplified [8, equation (5.19)], and the link between P6 and SD.I.a is

$$\begin{cases} y = \frac{x^2(x-1)^2}{4u(u-1)(u-x)}\left\{u' - \frac{u(u-1)}{x(x-1)}\right\}^2 + \frac{\Theta_\infty^2}{8}(1-2u) + \frac{\theta_0^2}{8}\left(1 - 2\frac{x}{u}\right) \\ \quad + \frac{\theta_1^2}{8}\left(2\frac{x-1}{u-1} - 1\right) + \frac{\theta_x^2}{8}\left(1 - 2\frac{x(u-1)}{u-x}\right), \\ \Theta_\infty = \theta_\infty + 1, \\ 2A_0 = \Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_x^2, \\ 4A_1 = -(\Theta_\infty^2 - \theta_0^2)(\theta_1^2 - \theta_x^2), \\ 4A_2 = -(\Theta_\infty^2 - \theta_x^2)(\theta_0^2 - \theta_1^2), \\ 4A_3 = (\Theta_\infty^2 - \theta_1^2)(\theta_0^2 - \theta_x^2), \\ 32A_4 = (\Theta_\infty^2 + \theta_x^2)(\theta_0^2 - \theta_1^2)^2 + (\Theta_\infty^2 - \theta_x^2)^2(\theta_0^2 + \theta_1^2). \end{cases} \tag{44}$$

The elimination of the intermediate constants (A_0, A_2, A_3, A_4) provides the link between, on one hand, the four essential parameters of the reduction (i.e. the two first integrals K_1, K_2 and the three constant phases β_j whose sum is zero) and, on the other hand, the four monodromy exponents $(\theta_\infty, \theta_0, \theta_1, \theta_x)$ of P6,

$$\begin{cases} 4K_1 = -\left[\beta_1^2 + \beta_2^2 + \beta_3^2 + \Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_x^2\right], \\ 48K_2 = -\frac{(\Theta_\infty^2 - \theta_0^2)(\theta_1^2 - \theta_x^2)}{\beta_1} - \frac{(\Theta_\infty^2 - \theta_x^2)(\theta_0^2 - \theta_1^2)}{\beta_2} + \frac{(\Theta_\infty^2 - \theta_1^2)(\theta_0^2 - \theta_x^2)}{\beta_3} \\ \quad + (\beta_1 - \beta_2)(\beta_2 - \beta_3)(\beta_3 - \beta_1), \\ \beta_1\beta_2\beta_3(\beta_1^2 + \beta_2^2 + \beta_3^2) + 2\beta_1\beta_2\beta_3(\Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_x^2) \\ \quad - 2\beta_1(\Theta_\infty^2\theta_0^2 + \theta_1^2\theta_x^2) - 2\beta_2(\Theta_\infty^2\theta_x^2 + \theta_0^2\theta_1^2) - 2\beta_3(\Theta_\infty^2\theta_1^2 + \theta_x^2\theta_0^2) = 0, \\ -(\Theta_\infty^2 - \theta_x^2)^2(\theta_0^2 - \theta_1^2)^2 - 2\beta_2^2\left[4\theta_1^4(\Theta_\infty^2 + \theta_x^2) + 4\theta_x^4(\theta_0^2 + \theta_1^2)\right. \\ \quad \left. + (\Theta_\infty^2 + \theta_x^2)(\theta_0^2 + \theta_1^2)(\Theta_\infty^2 + \theta_0^2 - 3\theta_1^2 - 3\theta_x^2)\right] - \beta_2^2\left[(\Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_x^2)^2\right. \\ \quad \left. - 2(\Theta_\infty^2 - \theta_x^2)(\theta_0^2 - \theta_1^2)\right] + 4\beta_2^3\beta_3(\Theta_\infty^2 - \theta_x^2)(\theta_0^2 - \theta_1^2) \\ \quad - 2\beta_2^4(\beta_2^2 + 2\beta_2\beta_3 + 2\beta_3^2)(\Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_x^2) - \beta_2^4(\beta_1^2 + \beta_3^2)^2 = 0. \end{cases} \tag{45}$$

The above first three equations are invariant under both the ternary symmetry on β_j and the quaternary symmetry on $(\Theta_\infty, \theta_0, \theta_1, \theta_x)$.

6. On the single valuedness of the six components

In the previous section, we have only proven that the variables ρ_j^2 and φ_j' are single valued (in this section, for brevity we use 'single valued' instead of 'with fixed critical singularities'). However, since the reduction is noncharacteristic, it remains to be proven that all the matrix

elements in the reduced Lax pair (19) are also single valued, so as to check the conjecture of Ablowitz, Ramani and Segur [2].

This question is quite similar to a much simpler one, which also seems to have never been investigated, so let us first solve this question in the simple case of the nonlinear Schrödinger equation,

$$\mathrm{i}A_t + pA_{xx} + q|A|^2 A = 0, \qquad pq \neq 0, \quad A \in \mathcal{C}, \quad (p, q) \in \mathcal{R}, \quad \mathrm{i}^2 = -1. \tag{46}$$

Its travelling wave reduction

$$A(x, t) = \sqrt{M(\xi)}\, \mathrm{e}^{\mathrm{i}(-\omega t + \varphi(\xi))}, \qquad \xi = x - ct, \tag{47}$$

admits the elliptic general solution

$$\begin{cases} M = -2\dfrac{p}{q}(\wp(\xi) - \wp(a)), \\ \varphi' = \dfrac{c}{2p} + \dfrac{\mathrm{j}}{2}\dfrac{\wp'(a)}{\wp(\xi) - \wp(a)}, \qquad \mathrm{j}^2 = -1, \\ \wp(a) = (4\omega p - c^2)/(12p^2), \end{cases} \tag{48}$$

in which $\wp(\xi)$ is the (even) elliptic function of Weierstrass and the arbitrary constants are the two elliptic invariants g_2, g_3 of the function $\wp$. Since $\wp'(a)$ is generically nonzero, the variable $\sqrt{M}$ is multivalued and behaves like $(\xi \pm a)^{1/2}$ near $\xi = \pm a$. However, the variables $\mathrm{e}^{\pm\mathrm{i}\arg A}$ present the same kind of branching, so a compensation occurs making the two fields A and $\bar{A}$ single valued. Indeed, the quadrature for φ is classical [3, section 18.7.3],

$$\wp'(a) \int \frac{\mathrm{d}\xi}{\wp(\xi) - \wp(a)} = 2\zeta(a)\xi + \log\sigma(\xi - a) - \log\sigma(\xi + a), \tag{49}$$

in which the meromorphic function ζ is the primitive of $-\wp$, the odd entire function $\sigma(z)$ behaves like z near $z = 0$ and the overall expressions of $\mathrm{e}^{\mathrm{i}\omega t}A$ and $\mathrm{e}^{-\mathrm{i}\omega t}\bar{A}$ in terms of ξ are indeed globally single valued (but not elliptic)

$$\begin{cases} \mathrm{e}^{\mathrm{i}\omega t}A = \sqrt{-\dfrac{2p}{q}}\sqrt{\wp(\xi) - \wp(a)}\, \mathrm{e}^{\mathrm{i}\mathrm{j}\zeta(a)\xi}\left(\dfrac{\sigma(\xi - a)}{\sigma(\xi + a)}\right)^{\mathrm{i}\mathrm{j}/2} \mathrm{e}^{\mathrm{i}c\xi/(2p)}, \qquad \mathrm{j}^2 = -1, \\ \mathrm{e}^{-\mathrm{i}\omega t}\bar{A} = \sqrt{-\dfrac{2p}{q}}\sqrt{\wp(\xi) - \wp(a)}\, \mathrm{e}^{-\mathrm{i}\mathrm{j}\zeta(a)\xi}\left(\dfrac{\sigma(\xi - a)}{\sigma(\xi + a)}\right)^{-\mathrm{i}\mathrm{j}/2} \mathrm{e}^{-\mathrm{i}c\xi/(2p)}. \end{cases} \tag{50}$$

Similarly, in the case of the three-wave system, the travelling wave reduction

$$\begin{cases} u_j(x, t) = c_j^{-1}\, \mathrm{e}^{\mathrm{i}(\beta_j t + \alpha\xi)}\psi_j(\xi), \\ \bar{u}_j(x, t) = c_j^{-1}\, \mathrm{e}^{-\mathrm{i}(\beta_j t + \alpha\xi)}\bar{\psi}_j(\xi), \\ \xi = ax + bt, \qquad (a, b) \neq (0, 0), \qquad \beta_1 + \beta_2 + \beta_3 = 0, \\ (ac_j + b)\dfrac{\mathrm{d}}{\mathrm{d}\xi}\psi_j = \mathrm{i}\bar{\psi}_k\bar{\psi}_l - \mathrm{i}\beta_j\psi_j, \\ (ac_j + b)\dfrac{\mathrm{d}}{\mathrm{d}\xi}\bar{\psi}_j = -\mathrm{i}\psi_k\psi_l + \mathrm{i}\beta_j\bar{\psi}_j \end{cases} \tag{51}$$

leads to an identical situation [7]

$$\begin{cases} \psi_j\bar{\psi}_j = b_j(\wp(\xi) - \wp(a_j)), \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}\arg\psi_j = \text{constant} + \dfrac{\mathrm{j}}{2}\dfrac{\wp'(a_j)}{\wp(\xi) - \wp(a_j)}, \qquad \mathrm{j}^2 = -1, \end{cases} \tag{52}$$

with an identical conclusion: single valuedness of $\psi_j(\xi)$ and $\bar{\psi}_j(\xi)$.

To come back to the reduction (9) to P6, establishing the single valuedness of $(c_j - \zeta)^{i\beta_j}\psi_j(\zeta)$ and $(c_j - \zeta)^{-i\beta_j}\bar{\psi}_j(\zeta)$ only requires extra care, the result being an expression similar to (50), in which the entire functions $\sigma(\xi - a)$, $\sigma(\xi + a)$ of Weierstrass are replaced by the two functions τ_1, τ_2 introduced by Painlevé

$$u = \pm x(x - 1)\, e^{-x}\theta_\infty^{-1} \frac{d}{dx}(\log \tau_1 - \log \tau_2), \tag{53}$$

and $u(x)$ obeys the P6 equation. These two functions τ_1, τ_2 have no movable singularities, but they have three fixed critical singularities, located at $x = \infty, 0, 1$.

7. Dual Lax pairs for the sixth Painlevé equation P6

The third-order monodromy matrix $\mathcal{M}$ of the reduced three-wave system, equation (19), admits in the complex μ plane the same singularities as another third-order matrix introduced [12] to describe the monodromy of a time-dependent Hamiltonian with three degrees of freedom, and later considered independently [18] from the point of view of its Laplace transform. The common singularities of this third-order monodromy matrix are $\mu = 0$ (of the Fuchsian type) and $\mu = \infty$ (of the non-Fuchsian type).

Moreover, a duality has been established by two different methods (factorization of a residue [12], Laplace transform in the μ space [18]) between the third-order Lax pair associated with the monodromy matrix and a second-order matrix Lax pair admitting as only singularities four Fuchsian points. This latter second-order Lax pair indeed admits the generic P6 equation as its zero-curvature condition. This should have two consequences. (i) There should exist an identification between the two systems (reduced three-wave, time-dependent Hamiltonian). (ii) The third-order matrix Lax pair, equation (19), should have a dual, second-order matrix Lax pair admitting P6 as its zero-curvature condition.

There exists a strong motivation to have a closer look at the resulting second-order Lax pair for P6; this is the hope that it might have a holomorphic dependence on the four monodromy exponents $(\theta_\infty, \theta_0, \theta_1, \theta_x)$, while the second-order matrix Lax pair of Jimbo and Miwa [13] has a meromorphic dependence on θ_∞. Indeed, P6 depends holomorphically on these exponents.

Let us first review the derivation of the second-order matrix Lax pair, then consider again the three-wave system.

7.1. Case of the three-degree-of-freedom Hamiltonian

The time-dependent Hamiltonian with three degrees of freedom [12, equation (3.56)]

$$H(q_j, p_j, x) = \tfrac{1}{4}[g_{31}(x)a_{13}a_{31} + g_{23}(x)a_{23}a_{32} + g_{12}(x)a_{12}a_{21}] \tag{54}$$

with the notation

$$
\begin{aligned}
a_{12} &= q_1 p_2 - q_2 p_1 + (\mu_1/q_1)q_2 + (\mu_2/q_2)q_1, \\
a_{21} &= q_2 p_1 - q_1 p_2 + (\mu_2/q_2)q_1 + (\mu_1/q_1)q_2, \\
a_{13} &= q_1 p_3 + q_3 p_1 - (\mu_1/q_1)q_3 + (\mu_3/q_3)q_1, \\
a_{31} &= q_3 p_1 + q_1 p_3 - (\mu_3/q_3)q_1 + (\mu_1/q_1)q_3, \\
a_{23} &= q_2 p_3 + q_3 p_2 - (\mu_2/q_2)q_3 + (\mu_3/q_3)q_2, \\
a_{32} &= q_3 p_2 + q_2 p_3 - (\mu_3/q_3)q_2 + (\mu_2/q_2)q_3
\end{aligned} \tag{55}
$$

generates a six-dimensional first-order system made of the six Hamilton equations in the canonical variables (q_j, p_j). For the following choice of the time-dependent coefficients:

$$g_{23} = [\log(g_2 - g_3)]', \qquad g_{31} = [\log(g_3 - g_1)]', \qquad g_{12} = 0, \qquad (g_2 - g_1)' = 0, \tag{56}$$

this system admits the time-independent first integral

$$I = a_{13}a_{31} + a_{23}a_{32} + a_{12}a_{21} + 2(\mu_1^2 + \mu_2^2 + \mu_3^2) \tag{57}$$

and the Lax pair [12, equations (3.62), (3.65)] (see also [18])

$$[\partial_x - \mathcal{L}_3, \partial_\lambda - \mathcal{M}_3] = 0,$$

$$\mathcal{L}_3 = -\lambda \begin{pmatrix} g_1' & 0 & 0 \\ 0 & g_2' & 0 \\ 0 & 0 & g_3' \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & g_{12}a_{12} & g_{31}a_{13} \\ g_{12}a_{21} & 0 & g_{23}a_{23} \\ g_{31}a_{31} & g_{23}a_{32} & 0 \end{pmatrix}$$

$$+ \frac{1}{2} \operatorname{diag}(g_{31}(\mu_1(q_3/q_1)^2 - \mu_3) + g_{12}(\mu_1(q_2/q_1)^2 - \mu_2),$$

$$g_{23}(\mu_2(q_3/q_2)^2 - \mu_3) + g_{12}(\mu_2(q_1/q_2)^2 - \mu_1), \tag{58}$$

$$g_{31}(\mu_3(q_1/q_3)^2 - \mu_1) + g_{23}(\mu_3(q_2/q_3)^2 - \mu_2)),$$

$$\mathcal{M}_3 = -\begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} + \frac{\mathcal{R}_{-1}}{\lambda}, \qquad \mathcal{R}_{-1} = \frac{1}{2}\begin{pmatrix} 2\mu_1 & a_{12} & a_{13} \\ a_{21} & 2\mu_2 & a_{23} \\ a_{31} & a_{32} & 2\mu_3 \end{pmatrix}.$$

Let us from now on choose the time-dependent coefficients as

$$g_1 = 0, \qquad g_2 = 1, \qquad g_3 = x, \qquad g_{23} = \frac{1}{x-1}, \qquad g_{31} = \frac{1}{x}, \qquad g_{12} = 0. \tag{59}$$

The residue $\mathcal{R}_{-1}$ has rank 2 and factorizes as [12]

$$\mathcal{R}_{-1} = FG, \tag{60}$$

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} q_1 & p_1 - \mu_1/q_1 \\ q_2 & p_2 - \mu_2/q_2 \\ q_3 & -p_3 + \mu_3/q_3 \end{pmatrix},$$

$$G = \frac{1}{\sqrt{2}} \begin{pmatrix} p_1 + \mu_1/q_1 & p_2 + \mu_2/q_2 & p_3 + \mu_3/q_3 \\ -q_1 & -q_2 & -q_3 \end{pmatrix}, \tag{61}$$

therefore, this third-order matrix Lax pair $(\mathcal{L}_3, \mathcal{M}_3)$ admits a dual, second-order matrix Lax pair $(\mathcal{L}_2, \mathcal{M}_2)$ defined as [12, equations (3.55), (3.61)]

$$[\partial_x - \mathcal{L}_2, \partial_\Lambda - \mathcal{M}_2] = 0, \tag{62}$$

$$\mathcal{L}_2 = -\frac{R_x}{\Lambda - x}, \qquad \mathcal{M}_2 = \frac{R_0}{\Lambda} + \frac{R_1}{\Lambda - 1} + \frac{R_x}{\Lambda - x}, \tag{63}$$

$$R_0 = -G\operatorname{diag}(1, 0, 0)F, \qquad R_1 = -G\operatorname{diag}(0, 1, 0)F,$$
$$R_x = -G\operatorname{diag}(0, 0, 1)F, \qquad R_\infty = -R_0 - R_1 - R_x = GF, \tag{64}$$

with the explicit expressions for the four residues

$$2R_\infty = \mu_1 + \mu_2 + \mu_3$$
$$+ \begin{pmatrix} q_1 p_1 + q_2 p_2 + q_3 p_3 & p_1^2 + p_2^2 - p_3^2 - (\mu_1/q_1)^2 - (\mu_2/q_2)^2 + (\mu_3/q_3)^2 \\ -q_1^2 - q_2^2 + q_3^2 & -q_1 p_1 - q_2 p_2 - q_3 p_3 \end{pmatrix}, \tag{65}$$

$$2R_0 = \begin{pmatrix} -q_1 p_1 & (\mu_1/q_1)^2 - p_1^2 \\ q_1^2 & q_1 p_1 \end{pmatrix} - \mu_1, \tag{66}$$

$$2R_1 = \begin{pmatrix} -q_2 p_2 & (\mu_2/q_2)^2 - p_2^2 \\ q_2^2 & q_2 p_2 \end{pmatrix} - \mu_2, \tag{67}$$

$$2R_x = \begin{pmatrix} -q_3 p_3 & -(\mu_3/q_3)^2 + p_3^2 \\ -q_3^2 & q_3 p_3 \end{pmatrix} - \mu_3. \tag{68}$$

The zero-curvature conditions of $(\mathcal{L}_2, \mathcal{M}_2)$ and $(\mathcal{L}_3, \mathcal{M}_3)$ are both equivalent to the Hamilton equations derived from (54). Therefore, since the singularities of the monodromy matrix $\mathcal{M}_2$ in the complex plane of Λ are four Fuchsian singularities (located at $\Lambda = \infty, 0, 1, x$), the Hamilton equations of (54) can be explicitly integrated in terms of P6 [12, 18]. In particular, the invariants of the four residues are constants of the motion,

$$\operatorname{tr} R_\infty = \mu_1 + \mu_2 + \mu_3, \qquad \operatorname{tr} R_0 = -\mu_1, \qquad \operatorname{tr} R_1 = -\mu_2, \qquad \operatorname{tr} R_x = -\mu_3, \tag{69}$$

$$\det R_\infty = -\frac{I}{4} + \frac{1}{2}(\mu_1 + \mu_2 + \mu_3)^2, \qquad \det R_0 = \det R_1 = \det R_x = 0. \tag{70}$$

The integration of the Hamilton equations is finally performed by identifying the coefficients of the matrix Lax pair (63) with the respective coefficients of a matrix Lax pair for the P6 equation (43). If one chooses for this Lax pair that in [13], the result is [12]

$$
\begin{cases}
q_1^2 + q_2^2 - q_3^2 = 0, \\
x q_1^2 - \left\{ (1+x) q_1^2 + x q_2^2 - q_3^2 \right\} u = 0, \\
q_1 p_1 + q_2 p_2 + q_3 p_3 - 2 a_0 = 0, \\
p_1^2 + p_2^2 - p_3^2 - (\mu_1/q_1)^2 - (\mu_2/q_2)^2 + (\mu_3/q_3)^2 = 0, \\
\dfrac{q_1 p_1}{u} + \dfrac{q_2 p_2}{u-1} + \dfrac{q_3 p_3}{u-x} + \dfrac{1}{u-x} - x(x-1) u' = 0, \\
\dfrac{x(x-1) u'}{u(u-1)(u-x)} + 2v + \dfrac{\mu_1}{u} + \dfrac{\mu_2}{u-1} + \dfrac{\mu_3}{u-x} = 0, \\
\dfrac{x(x-1) v'}{u(u-1)(u-x)} + \left\{ \dfrac{1}{u} + \dfrac{1}{u-1} + \dfrac{1}{u-x} \right\} v^2 \\
\quad + \left\{ \dfrac{\mu_1}{u-1} + \dfrac{\mu_1}{u-x} + \dfrac{\mu_2}{u-x} + \dfrac{\mu_2}{u} + \dfrac{\mu_3+1}{u} + \dfrac{\mu_3+1}{u-1} \right\} v \\
\quad + \dfrac{\mu_1^2 + \mu_2^2 + \mu_3^2 + 2(\mu_1 + \mu_2 + \mu_3) - 4 a_0^2 - 4 a_0}{4 u(u-1)(u-x)} = 0, \\
\theta_\infty^2 = (2 a_0 + 1)^2, \qquad \theta_0^2 = \mu_1^2, \qquad \theta_1^2 = \mu_2^2, \qquad \theta_x^2 = \mu_3^2, \qquad I = 8 a_0^2.
\end{cases}
\tag{71}
$$

7.2. Case of the three-wave system

In order to perform the identification of the Hamiltonian system (54) with the three-wave reduced system (10), several methods are possible.

A first method is to factorize the residue $\mathcal{M}_{-1}$ into a product similar to (61). Since this residue has rank 3, one first lowers its rank to 2 by applying the transition matrix $P = \mu^a \mathcal{I}$ to the Lax pair, in which $\mathcal{I}$ is the identity matrix and a is any of the constant roots of the characteristic polynomial (21). The factorization of the resulting rank 2 matrix as

$$R = \mathcal{M}_{-1} - a = F G, \qquad \operatorname{tr} \mathcal{M}_{-1} = 0, \tag{72}$$

with F a (3, 2) matrix and G a (2, 3) matrix, both of rank 2, is possible [19, section 3.5.4] but it is not unique. In particular, if the elements of F and G are restricted to rational functions

of R'_{ij} s, the resulting elements of F and G depend on four arbitrary functions of R'_{ij} s, with no specific direct criterion to choose them; therefore, this is probably not the good method. However, with definitions (63) and (64), the invariants of the four residues are independent of the choice of the four gauges,

$$\operatorname{tr} R_\infty = 3a, \qquad \operatorname{tr} R_0 = R_{11}, \qquad \operatorname{tr} R_1 = R_{22}, \qquad \operatorname{tr} R_x = R_{33}, \quad (73)$$

$$\det R_\infty = 3a^2 + Q_2, \qquad \det R_0 = \det R_1 = \det R_x = 0, \quad (74)$$

$$a^3 + Q_2 a + Q_3 = 0. \quad (75)$$

A second method consists of identifying the invariants of the two residues of the third-order matrix Lax pairs. Since the residue $\mathcal{R}_{-1}$ is not traceless, this identification is

$$\forall z : \det(\mathcal{M}_{-1} - z) = \det\left(\mathcal{R}_{-1} - \frac{\mu_1 + \mu_2 + \mu_3}{3} - z\right), \quad (76)$$

i.e.

$$\begin{cases} K_1 + \dfrac{\beta_1^2 + \beta_2^2 + \beta_3^2}{6} = -\dfrac{I}{4} + \dfrac{(\mu_1 + \mu_2 + \mu_3)^2}{6}, \\[2mm] K_2 - \dfrac{(\beta_2 - \beta_3)(\beta_3 - \beta_1)(\beta_1 - \beta_2)}{54} = -i\dfrac{(\mu_1 + \mu_2 + \mu_3)I}{24} + \dfrac{5i}{108}(\mu_1 + \mu_2 + \mu_3)^3. \end{cases} \quad (77)$$

One difference between the two systems is the nature of the involved constants. The Hamiltonian system has three fixed constants (μ_1, μ_2, μ_3) and one movable constant (the first integral I), while the reduced three-wave system has two fixed constants (two elements among the three β_j) and two movable constants (the two first integrals K_1, K_2).

8. Conclusion

The problem of factorizing the residue R in (72) is still open and currently under investigation. If it were solved and if the resulting second-order matrix Lax pair for P6 were holomorphic in the four monodromy exponents, this would be an improvement over the second-order matrix Lax pair of Jimbo and Miwa [13], which has a meromorphic dependence on Θ_∞. For a comparative discussion of the Lax pairs of P6, see e.g. [16].

Another direction of research could be to try to match the four-fold symmetry of P6 with the N-fold symmetry of the reduced N-wave system. In the case $N = 3$ considered in this paper, the correspondence between the reduced three-wave and P6 involves $(\Theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2)$, see (45), and not $(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2)$ (as is the case in e.g. the second-order scalar Lax pair of Fuchs [10]), i.e. it contains the shift $\Theta_\infty = \theta_\infty + 1$. This discrepancy could disappear with the four-wave system, $N = 4$.

Finally, let us mention (and we thank the referee for signalling this reference) a different approach [14] which also exhibits, in a more general framework, a relationship between the 3WRI and P6 and, thanks to some freedom which allows the creation of suitable zero elements in the third-order ODE Lax pair, is able to perform a projection on a second-order Lax pair such as (63).

Acknowledgments

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